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Bethe ansatz for an AdS/CFT open spin chain with non-diagonal boundaries

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Abstract

We consider the integrable open-chain transfer matrix corresponding to a $Y = 0$ brane at one boundary, and a $Y_\theta = 0$ brane (rotated with the respect to the former by an angle θ) at the other boundary. We determine the exact eigenvalues of this transfer matrix in terms of solutions of a corresponding set of Bethe equations.

1 Introduction

Two remarkable conjectures have commanded considerable attention for over a decade: the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence [1, 2], positing the equivalence of type IIB superstring theory on $AdS_5 \times S^5$ [3] and $\mathcal{N}=4$ supersymmetric $SU(N)$ Yang-Mills theory in 3+1 dimensions [4]; and the integrability of the spectral problem in planar $\mathrm{AdS}_5/\mathrm{CFT}_4$ [5], positing that the energies of string states, or equivalently, the scaling dimensions of all local gauge-invariant operators in the planar limit of the dual gauge theory, are described by an integrable 1+1 dimensional model.[1] We shall refer to the latter as the AdS/CFT integrable model.

This AdS/CFT integrable model is essentially the string world-sheet quantum field theory in a light-cone gauge. It has a centrally-extended $su(2|2)$ symmetry [2], and the spectrum includes four fundamental particles: two bosons, and two fermions. The exact (non-relativistic) dispersion relation is known, as are exact bulk and boundary world-sheet S-matrices.

The momentum quantization condition for a set of such particles on a ring (i.e., periodic boundary conditions) of finite length leads [8, 9] to the all-loop asymptotic Bethe equations [10], which determine the energies of closed strings/scaling dimensions of single-trace operators in the dual gauge theory, up to wrapping (finite-size) corrections.

Similarly, the momentum quantization condition for a set of such particles on an *interval* of finite length leads to all-loop asymptotic Bethe equations that determine the energies of *open* strings/scaling dimensions of *determinant-like* operators in the dual gauge theory, again up to wrapping (finite-size) corrections. The detailed results depend on the specific boundary conditions at the two ends of the interval. Among the integrable cases that have been studied are $Y=0$ branes [11, 12] at both ends [13, 14, 15]; and $Y=0$ at one end and $\bar{Y}=0$ at the other end [16]. (For a review of integrable boundary conditions in AdS/CFT, see [17].)

The key technical step in deriving the various asymptotic Bethe equations is to determine the eigenvalues of the corresponding integrable inhomogeneous transfer matrices, which are constructed with the bulk and – for cases with boundaries – boundary S-matrices. The boundary S-matrices for $Y=0$ and $\bar{Y}=0$ branes are diagonal. However, the boundary S-matrix for a $Y_\theta = 0$ brane [16], which interpolates between them, is not diagonal. Hence, the problem of diagonalizing the transfer matrix constructed with the latter boundary S-matrix is nontrivial, and is the main goal of this paper. Our strategy is to exploit the unbroken $u(1)$ symmetry by carrying out the first step of the nested algebraic Bethe ansatz, following [18, 19]. This leads to a second-level open-chain spin-1/2 XXX transfer matrix with non-diagonal boundary terms, which we diagonalize by introducing an inhomogeneous term in its T-Q equation [20, 21]. A similar strategy was employed to solve the open Hubbard [22] and supersymmetric t-J [23] models with non-diagonal boundary interactions; however, those works used coordinate Bethe ansatz (instead of nested algebraic Bethe ansatz) for the first step.

The paper is organized as follows. In section 2 we introduce our notations, recall the

[1]Integrability is believed to appear also for $\mathrm{AdS}_4/\mathrm{CFT}_3$ [6] and $\mathrm{AdS}_3/\mathrm{CFT}_2$ [7]. However, we focus here on $\mathrm{AdS}_5/\mathrm{CFT}_4$, which is the simplest and best-understood case.

[2]Actually, the symmetry consists of two copies of this algebra, but we focus here on just one copy.

relevant AdS/CFT bulk and boundary S-matrices, and review the construction of the corresponding integrable open-chain transfer matrix. In section 3 we determine the exact eigenvalues of this transfer matrix in terms of solutions of a corresponding set of Bethe equations. We then use the unbroken $su(2)$ symmetry to derive formulas for the number of distinct eigenvalues (and hence, number of solutions of the Bethe equations) and their degeneracies. We check these results numerically for small system size. In section 4 we briefly discuss our results and note some remaining problems. In the appendix we propose a generating functional for the eigenvalues of transfer matrices whose auxiliary spaces belong to higher-dimensional representations of $su(2|2)$.

2 Construction of the transfer matrix

Here we introduce our notations, recall the relevant AdS/CFT bulk and boundary S-matrices, and review the construction of the corresponding integrable open-chain transfer matrix.

2.1 Parametrization

Following Arutyunov and Frolov [24], we use the elliptic parametrization for the momentum p and the parameters $x^{\pm}$ [3]

$$p(z) = 2\,\mathrm{am}(z,k)\,, \qquad x^{\pm}(z) = \frac{1}{2g}\left(\frac{\mathrm{cn}(z,k)}{\mathrm{sn}(z,k)} \pm i\right)(1+\mathrm{dn}(z,k))\,, \qquad k=-4g^2\,, \tag{2.1}$$

such that

$$\frac{x^+}{x^-} = e^{ip}\,, \tag{2.2}$$

and

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{2i}{g}\,. \tag{2.3}$$

(We shall often refrain from exhibiting the dependence of $x^{\pm}$ and p on the uniformizing parameter z.) The parameter g is the coupling constant of the AdS/CFT integrable model (string tension), which is related to the 't Hooft coupling λ of the dual gauge theory by $g = \sqrt{\lambda}/(2\pi) > 0$.

The two periods are given by

$$2\omega_1 = 4\mathrm{K}(k)\,, \qquad 2\omega_2 = 4i\mathrm{K}(1-k) - 4\mathrm{K}(k)\,, \tag{2.4}$$

[3] We generally follow the conventions in *Mathematica* for the Jacobi elliptic functions, e.g. $\mathrm{cn}(z,k) =$ JacobiCN$[z,k]$ and $\mathrm{K}(k) =$ EllipticK$[k]$. The one exception is $\mathrm{am}(z,k) = -i$Log[JacobiCN$[z,k]$ + iJacobiSN$[z,k]$], which is consistent with both (2.2) and (2.6).

where $\mathrm{K}(k)$ is the complete elliptic integral of the first kind. The crossing transformation is effectuated with a shift of z by the half-period ω_2,

$$x^{\pm}(z+\omega_2) = \frac{1}{x^{\pm}(z)}\,, \tag{2.5}$$

$$p(z+\omega_2) = -p(z)\,, \tag{2.6}$$

$$E(z+\omega_2) = -E(z)\,, \tag{2.7}$$

where $E(z) = -\frac{ig}{2}\left(x^+ - \frac{1}{x^+} - x^- + \frac{1}{x^-}\right) = \mathrm{dn}(z,k)$ is the energy. We note that $z \mapsto -z$ corresponds to a reflection,

$$x^{\pm}(-z) = -x^{\mp}(z)\,, \qquad p(-z) = -p(z)\,, \qquad E(-z) = E(z)\,. \tag{2.8}$$

We define $u(z)$ by

$$x^{\pm} + \frac{1}{x^{\pm}} = \frac{2}{g}\left(u \pm \frac{i}{2}\right)\,, \tag{2.9}$$

and therefore

$$u(z+\omega_2) = u(z)\,, \qquad u(-z) = -u(z)\,. \tag{2.10}$$

2.2 S-matrices

As already noted, there are four fundamental particles. Let us denote the corresponding Zamolodchikov-Faddeev operators by $A_i^\dagger(z)\,, i = 1,2,3,4$, where $i = 1,2$ are bosonic and $i = 3,4$ are fermionic. The matrix elements of the bulk S-matrix are defined by

$$A_i^\dagger(z_1)\, A_j^\dagger(z_2) = \sum_{i',j'=1}^{4} S_{i\,j}^{i'j'}(z_1,z_2)\, A_{j'}^\dagger(z_2)\, A_{i'}^\dagger(z_1)\,, \tag{2.11}$$

which can be arranged into a 16×16 matrix as follows

$$S(z_1,z_2) = \sum_{i,i',j,j'=1}^{4} S_{i\,j}^{i'j'}(z_1,z_2)\, e_{i\,i'} \otimes e_{j\,j'}\,, \qquad (e_{ij})_{ab} = \delta_{a,i}\delta_{b,j}\,. \tag{2.12}$$

We work with a graded version of Beisert's $su(2|2)$ S-matrix [25]. Specifically, following Arutyunov and Frolov [26], we take

$$S(z_1,z_2) = \sum_{k=1}^{10} a_k(z_1,z_2)\Lambda_k\,, \tag{2.13}$$

where the matrices $\Lambda_1\,, \ldots\,, \Lambda_{10}$ are given in terms of quantities E_{kilj} defined by

$$E_{kilj} = e_{ki} \otimes e_{lj}\,. \tag{2.14}$$

Hence, $S(z_1, z_2)$ has the following matrix structure

$$
\left(\begin{array}{cccccccccccccccc}
a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{a_1}{2}+\frac{a_2}{2} & 0 & 0 & \frac{a_1}{2}-\frac{a_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & a_7 & 0 & 0 & -a_7 & 0 \\
0 & 0 & a_5 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 & 0 \\
0 & \frac{a_1}{2}-\frac{a_2}{2} & 0 & 0 & \frac{a_1}{2}+\frac{a_2}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -a_7 & 0 & 0 & a_7 & 0 \\
0 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_5 & 0 & 0 & a_9 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_5 & 0 & 0 & 0 & 0 & 0 & a_9 & 0 & 0 \\
0 & 0 & a_{10} & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & a_{10} & 0 & 0 & a_6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 & 0 \\
0 & a_8 & 0 & 0 & -a_8 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_3}{2}+\frac{a_4}{2} & 0 & 0 & \frac{a_3}{2}-\frac{a_4}{2} & 0 \\
0 & 0 & 0 & a_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{10} & 0 & 0 & 0 & 0 & 0 & a_6 & 0 & 0 \\
0 & -a_8 & 0 & 0 & a_8 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_3}{2}-\frac{a_4}{2} & 0 & 0 & \frac{a_3}{2}+\frac{a_4}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_3
\end{array}\right)
$$

and the matrix elements $a_k = a_k(z_1, z_2)$ are given by [26]

$$
\begin{aligned}
a_1 &= 1\,, \\
a_2 &= 2\,\frac{(x_1^+ - x_2^+)(x_1^- x_2^+ - 1)x_2^-}{(x_1^+ - x_2^-)(x_1^- x_2^- - 1)x_2^+} - 1\,, \\
a_3 &= \frac{x_2^+ - x_1^-}{x_2^- - x_1^+}\frac{\tilde{\eta}_1\tilde{\eta}_2}{\eta_1\eta_2}\,, \\
a_4 &= \frac{(x_1^- - x_2^+)}{(x_2^- - x_1^+)}\frac{\tilde{\eta}_1\tilde{\eta}_2}{\eta_1\eta_2} - 2\,\frac{(x_2^- x_1^+ - 1)(x_1^+ - x_2^+)x_1^-}{(x_1^- x_2^- - 1)(x_2^- - x_1^+)x_1^+}\frac{\tilde{\eta}_1\tilde{\eta}_2}{\eta_1\eta_2}\,, \\
a_5 &= \frac{x_2^- - x_1^-}{x_2^- - x_1^+}\frac{\tilde{\eta}_2}{\eta_2}\,, \\
a_6 &= \frac{x_1^+ - x_2^+}{x_1^+ - x_2^-}\frac{\tilde{\eta}_1}{\eta_1}\,, \\
a_7 &= -\frac{i(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{(x_1^- x_2^- - 1)(x_2^- - x_1^+)}\frac{1}{\eta_1\eta_2}\,, \\
a_8 &= \frac{i x_1^- x_2^- (x_1^+ - x_2^+)}{(x_1^- x_2^- - 1)(x_2^- - x_1^+)x_1^+ x_2^+}\tilde{\eta}_1\tilde{\eta}_2\,, \\
a_9 &= \frac{x_1^+ - x_1^-}{x_1^+ - x_2^-}\frac{\tilde{\eta}_2}{\eta_1}\,, \\
a_{10} &= \frac{x_2^- - x_2^+}{x_2^- - x_1^+}\frac{\tilde{\eta}_1}{\eta_2}\,,
\end{aligned}
\qquad (2.15)
$$

where $x_i^{\pm} = x^{\pm}(z_i)$. Moreover,

$$\eta_1 = e^{ip_2/2}\eta(z_1)\,, \quad \eta_2 = \eta(z_2)\,, \quad \tilde{\eta}_1 = \eta(z_1)\,, \quad \tilde{\eta}_2 = e^{ip_1/2}\eta(z_2)\,, \tag{2.16}$$

where $p_i = p(z_i)$ and

$$\eta(z) = \sqrt{\frac{2}{g}}\frac{\text{dn }\frac{z}{2}\big(\text{ cn }\frac{z}{2} + i \text{ sn }\frac{z}{2}\text{ dn }\frac{z}{2}\big)}{1+4g^2\,\text{sn}^4\,\frac{z}{2}}\,. \tag{2.17}$$

This S-matrix satisfies the graded Yang-Baxter equation [4]

$$S_{12}(z_1,z_2)\,S_{13}(z_1,z_3)\,S_{23}(z_2,z_3) = S_{23}(z_2,z_3)\,S_{13}(z_1,z_3)\,S_{12}(z_1,z_2)\,, \tag{2.18}$$

where $S_{12}(z_1,z_2) = S(z_1,z_2)\otimes\mathbb{I}\,, S_{13}(z_1,z_3) = \mathcal{P}_{23}S_{12}(z_1,z_3)\mathcal{P}_{23}\,, S_{23}(z_2,z_3) = \mathcal{P}_{12}S_{13}(z_2,z_3)\mathcal{P}_{12}$, and $\mathcal{P}$ denotes the *graded* permutation matrix

$$\mathcal{P} = \sum_{a,b=1}^{4}(-1)^{\epsilon_a\epsilon_b}e_{ab}\otimes e_{ba}\,, \tag{2.19}$$

where the gradings are given by $\epsilon_1 = \epsilon_2 = 0\,, \epsilon_3 = \epsilon_4 = 1$. As is well known, the S-matrix has $su(2)\oplus su(2)$ symmetry,

$$[S_{12}(z_1,z_2)\,, G_1\,G_2] = 0\,, \qquad G = \begin{pmatrix} g_L & 0 \\ 0 & g_R \end{pmatrix}\,, \tag{2.20}$$

where g_L and g_R are independent 2×2 special unitary matrices.

For the right boundary, we consider a boundary S-matrix that is diagonal [12, 27],

$$R^R(z) = \text{diag}(e^{-ip/2}\,, -e^{ip/2}\,, 1\,, 1)\,, \tag{2.21}$$

corresponding to a $Y=0$ brane [11, 12]. It satisfies the right boundary Yang-Baxter equation [28, 29, 30]

$$S_{12}(z_1,z_2)\,R_1^R(z_1)\,S_{21}(z_2,-z_1)\,R_2^R(z_2) = R_2^R(z_2)\,S_{12}(z_1,-z_2)\,R_1^R(z_1)\,S_{21}(-z_2,-z_1)\,, \tag{2.22}$$

where $S_{21}(z_1,z_2) = \mathcal{P}_{12}\,S_{12}(z_1,z_2)\,\mathcal{P}_{12}$, $R_1^R(z) = R^R(z)\otimes\mathbb{I}$ and $R_2^R(z) = \mathcal{P}_{12}\,R_1^R(z)\,\mathcal{P}_{12}$.

For the left boundary, we consider a non-diagonal boundary S-matrix [16],

$$R^L(z) = O^t(\theta)R^R(-z)O(\theta)\,, \tag{2.23}$$

where $O(\theta)$ is the rotation matrix

$$O(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\,, \tag{2.24}$$

and θ is an arbitrary angle. This boundary S-matrix, which corresponds to a $Y_\theta = 0$ brane, interpolates between $Y=0$ ($\theta=0$) and $\bar{Y}=0$ ($\theta=\pi/2$). This boundary S-matrix evidently preserves the right $su(2)$ symmetry

$$\big[R^L(z)\,, g_R\big] = 0\,, \tag{2.25}$$

but breaks the left $su(2)$ symmetry.

[4]We work in the so-called string (rather than spin-chain) frame/basis, where the S-matrix obeys a standard (rather than twisted) Yang-Baxter equation.

2.3 Transfer matrix

The open-chain transfer matrix for a single copy of $su(2|2)$ is given by [29, 31, 32] [5]

$$t(z\,;\{z_i\}) = \mathrm{str}_a \left\{ R^L_a(z)\, T_a(z\,;\{z_i\})\, R^R_a(z)\, \widehat{T}_a(z\,;\{z_i\}) \right\} \,, \tag{2.26}$$

where the monodromy matrices are given by

$$\begin{aligned} T_a(z\,;\{z_i\}) &= S_{aN}(z, z_N) \cdots S_{a1}(z, z_1)\,, \\ \widehat{T}_a(z\,;\{z_i\}) &= S_{1a}(z_1, -z) \cdots S_{Na}(z_N, -z)\,, \end{aligned} \tag{2.27}$$

the auxiliary space is denoted by a, and str denotes super trace. The $\{z_i\}$, which correspond to the rapidities of the N particles on an interval, are to be regarded as fixed inhomogeneities. (To lighten the notation, we shall often suppress the dependence on these inhomogeneities.) By construction (see e.g. [29, 31]), the transfer matrix has the fundamental commutativity property

$$[t(z\,;\{z_i\})\,, t(z'\,;\{z_i\})] = 0 \tag{2.28}$$

for arbitrary values of z and z'. For the boundary S-matrices (2.21) and (2.23) that we consider here, the transfer matrix also has the right $su(2)$ symmetry

$$\left[t(z\,;\{z_i\})\,, \vec{S} \right] = 0\,, \tag{2.29}$$

where

$$\vec{S} = \sum_{n=1}^{N} \vec{S}_n\,, \qquad \vec{S}_n = \left(\begin{array}{cc} 0 & 0 \\ 0 & \frac{1}{2}\vec{\sigma} \end{array} \right)_n \,. \tag{2.30}$$

3 Exact diagonalization of the transfer matrix

We turn now to the main task of deriving the eigenvalues of the transfer matrix (2.26) and obtaining the corresponding Bethe equations.

3.1 Nested algebraic Bethe ansatz

The transfer matrix has an unbroken $u(1) \subset su(2)$ symmetry. In particular, the state with "all spins down"

$$|0\rangle = \otimes_{j=1}^{N} |0\rangle_j\,, \qquad |0\rangle_j = \left(\begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \right)_j \tag{3.1}$$

[5] In order to derive the AdS/CFT all-loop asymptotic Bethe equations, one must also take into account the second copy of the $su(2|2)$ S-matrix. However, the most difficult technical part of the derivation is the diagonalization of the transfer matrix for a single copy, on which we focus here.

is an eigenstate of the transfer matrix. Therefore, using this state as the reference state, we can carry out the first step of the nested algebraic Bethe ansatz, following [18, 19]. To this end, it is convenient to write the boundary S-matrices (2.21), (2.23) as

$$R^R(z) = \begin{pmatrix} K_1^-(z) & & & \\ & K_2^-(z) & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad R^L(z) = \begin{pmatrix} K_1^+(z) & K_2^+(z) & & \\ K_3^+(z) & K_4^+(z) & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \tag{3.2}$$

where

$$\begin{aligned} K_1^-(z) &= e^{-ip(z)/2}, \quad K_2^-(z) = -e^{ip(z)/2}, \\ K_1^+(z) &= \cos^2\theta e^{ip(z)/2} - \sin^2\theta e^{-ip(z)/2}, \quad K_2^+(z) = \sin\theta\cos\theta(e^{ip(z)/2} + e^{-ip(z)/2}), \\ K_3^+(z) &= \sin\theta\cos\theta(e^{ip(z)/2} + e^{-ip(z)/2}), \quad K_4^+(z) = \sin^2\theta e^{ip(z)/2} - \cos^2\theta e^{-ip(z)/2}. \end{aligned} \tag{3.3}$$

We also write the monodromy matrices (2.27) as follows

$$T_a(z\,;\{z_i\}) = \begin{pmatrix} A_{11}(z) & A_{12}(z) & E_1(z) & C_1(z) \\ A_{21}(z) & A_{22}(z) & E_2(z) & C_2(z) \\ C_4(z) & C_5(z) & D(z) & C_3(z) \\ B_1(z) & B_2(z) & F(z) & B(z) \end{pmatrix}, \tag{3.4}$$

$$\widehat{T}_a(z\,;\{z_i\}) = \begin{pmatrix} \bar{A}_{11}(z) & \bar{A}_{12}(z) & \bar{E}_1(z) & \bar{C}_1(z) \\ \bar{A}_{21}(z) & \bar{A}_{22}(z) & \bar{E}_2(z) & \bar{C}_2(z) \\ \bar{C}_4(z) & \bar{C}_5(z) & \bar{D}(z) & \bar{C}_3(z) \\ \bar{B}_1(z) & \bar{B}_2(z) & \bar{F}(z) & \bar{B}(z) \end{pmatrix}. \tag{3.5}$$

3.1.1 The action of the transfer matrix on the reference state

We observe that the elements of the monodromy matrices have the following action on the reference state

$$A_{11}(z)|0\rangle = A_{22}(z)|0\rangle = \prod_{i=1}^{N} a_5(z, z_i)|0\rangle\,, \tag{3.6}$$

$$D(z)|0\rangle = \prod_{i=1}^{N} a_{14}(z, z_i)|0\rangle, \quad B(z)|0\rangle = \prod_{i=1}^{N} a_3(z, z_i)|0\rangle\,, \tag{3.7}$$

$$A_{12}(z)|0\rangle = A_{21}(z)|0\rangle = 0, \quad C_j(z)|0\rangle = 0\,, \tag{3.8}$$

$$F(z)|0\rangle \neq 0, \quad B_j(z)|0\rangle \neq 0\,, \tag{3.9}$$

$$\bar{A}_{11}(z)|0\rangle = \bar{A}_{22}(z)|0\rangle = \prod_{i=1}^{N} a_6(z_i, -z)|0\rangle, \tag{3.10}$$

$$\bar{D}(z)|0\rangle = \prod_{i=1}^{N} a_{14}(z_i, -z)|0\rangle, \quad \bar{B}(z)|0\rangle = \prod_{i=1}^{N} a_3(z_i, -z)|0\rangle\,, \tag{3.11}$$

$$\bar{A}_{12}(z)|0\rangle = \bar{A}_{21}(z)|0\rangle = 0, \quad \bar{C}_j(z)|0\rangle = 0\,, \tag{3.12}$$

$$\bar{F}(z)|0\rangle \neq 0, \quad \bar{B}_j(z)|0\rangle \neq 0\,. \tag{3.13}$$

Here and below we use the following notations

$$a_{11}(z_1, z_2) = \tfrac{1}{2}(a_1(z_1, z_2) - a_2(z_1, z_2))\,, \qquad a_{12}(z_1, z_2) = \tfrac{1}{2}(a_1(z_1, z_2) + a_2(z_1, z_2))\,, \tag{3.14}$$

$$a_{13}(z_1, z_2) = \tfrac{1}{2}(a_3(z_1, z_2) - a_4(z_1, z_2))\,, \qquad a_{14}(z_1, z_2) = \tfrac{1}{2}(a_3(z_1, z_2) + a_4(z_1, z_2))\,, \tag{3.15}$$

where $a_1\,, \ldots\,, a_{10}$ are given by (2.15).

The double-row monodromy matrix is defined as

$$U_a(z) = T_a(z\,; \{z_i\})\, R_a^R(z)\, \widehat{T}_a(z\,; \{z_i\}) = \begin{pmatrix} \mathcal{A}_{11}(z) & \mathcal{A}_{12}(z) & \mathcal{E}_1(z) & \mathcal{C}_1(z) \\ \mathcal{A}_{21}(z) & \mathcal{A}_{22}(z) & \mathcal{E}_2(z) & \mathcal{C}_2(z) \\ \mathcal{C}_4(z) & \mathcal{C}_5(z) & \mathcal{D}(z) & \mathcal{C}_3(z) \\ \mathcal{B}_1(z) & \mathcal{B}_2(z) & \mathcal{F}(z) & \mathcal{B}(z) \end{pmatrix}, \tag{3.16}$$

We obtain

$$
\begin{aligned}
\mathcal{B}(z)|0\rangle &= B(z)\bar{B}(z)|0\rangle, &(3.17)\\
\mathcal{A}_{11}(z)|0\rangle &= K_1^-(z)A_{11}(z)\bar{A}_{11}(z)|0\rangle + C_1(z)\bar{B}_1(z)|0\rangle\,, &(3.18)\\
\mathcal{A}_{22}(z)|0\rangle &= K_2^-(z)A_{22}(z)\bar{A}_{22}(z)|0\rangle + C_2(z)\bar{B}_2(z)|0\rangle\,, &(3.19)\\
\mathcal{A}_{12}(z)|0\rangle &= C_1(z)\bar{B}_2(z)|0\rangle\,, &(3.20)\\
\mathcal{A}_{21}(z)|0\rangle &= C_2(z)\bar{B}_1(z)|0\rangle\,, &(3.21)\\
\mathcal{D}(z)|0\rangle &= K_1^-(z)C_4(z)\bar{E}_1(z)|0\rangle + K_2^-(z)C_5(z)\bar{E}_2(z)|0\rangle + C_3(z)\bar{F}(z)|0\rangle \\
&\quad + D(z)\bar{D}(z)|0\rangle\,. &(3.22)
\end{aligned}
$$

In order to obtain the actions of operators $\mathcal{A}_{ij}(z)$ and $\mathcal{D}(z)$ on the reference state, we use exchange relations derived from the Yang-Baxter equation (2.18)

$$
T_1(z\,;\{z_i\})\,S_{12}(z,-z)\,\widehat{T}_2(z\,;\{z_i\}) = \widehat{T}_2(z\,;\{z_i\})\,S_{12}(z,-z)\,T_1(z\,;\{z_i\})\,. \tag{3.23}
$$

After some algebra, we obtain

$$
\begin{aligned}
C_1(z)\bar{B}_1(z)|0\rangle &= \frac{a_9(z,-z)}{a_3(z,-z)}\left[A_{11}(z)\bar{A}_{11}(z) - B(z)\bar{B}(z)\right]|0\rangle\,, &(3.24)\\
C_2(z)\bar{B}_2(z)|0\rangle &= \frac{a_9(z,-z)}{a_3(z,-z)}\left[A_{22}(z)\bar{A}_{22}(z) - B(z)\bar{B}(z)\right]|0\rangle\,, &(3.25)\\
\bar{C}_1(z)B_1(z)|0\rangle &= \frac{a_{10}(z,-z)}{a_3(z,-z)}\left[A_{11}(z)\bar{A}_{11}(z) - B(z)\bar{B}(z)\right]|0\rangle\,, &(3.26)\\
\bar{C}_2(z)B_2(z)|0\rangle &= \frac{a_{10}(z,-z)}{a_3(z,-z)}\left[A_{22}(z)\bar{A}_{22}(z) - B(z)\bar{B}(z)\right]|0\rangle\,, &(3.27)\\
\mathcal{A}_{12}(z)|0\rangle &= C_1(z)\bar{B}_2(z)|0\rangle = 0\,, &(3.28)\\
\mathcal{A}_{21}(z)|0\rangle &= C_2(z)\bar{B}_1(z)|0\rangle = 0\,, &(3.29)\\
\mathcal{A}_{11}(z)|0\rangle &= \left[K_1^-(z) + \frac{a_9(z,-z)}{a_3(z,-z)}\right]A_{11}(z)\bar{A}_{11}(z)|0\rangle - \frac{a_9(z,-z)}{a_3(z,-z)}B(z)\bar{B}(z)|0\rangle\,, &(3.30)\\
\mathcal{A}_{22}(z)|0\rangle &= \left[K_2^-(z) + \frac{a_9(z,-z)}{a_3(z,-z)}\right]A_{22}(z)\bar{A}_{22}(z)|0\rangle - \frac{a_9(z,-z)}{a_3(z,-z)}B(z)\bar{B}(z)|0\rangle\,. &(3.31)
\end{aligned}
$$

We define

$$
\tilde{\mathcal{A}}_{ij}(z) = \mathcal{A}_{ij}(z) + \delta_{ij}\frac{a_9(z,-z)}{a_3(z,-z)}\mathcal{B}(z)\,. \tag{3.32}
$$

Then

$$\begin{aligned}
\tilde{\mathcal{A}}_{12}(z)|0\rangle &= \tilde{\mathcal{A}}_{21}(z)|0\rangle = 0\,, &(3.33)\\
\tilde{\mathcal{A}}_{11}(z)|0\rangle &= \left[K_1^-(z) + \frac{a_9(z,-z)}{a_3(z,-z)}\right] A_{11}(z)\bar{A}_{11}(z)|0\rangle \\
&= \frac{e^{ip(z)/2}+e^{-ip(z)/2}}{2}\prod_{k=1}^{N} a_5(z,z_k)a_6(z_k,-z)|0\rangle\,, &(3.34)\\
\tilde{\mathcal{A}}_{22}(z)|0\rangle &= \left[K_2^-(z) + \frac{a_9(z,-z)}{a_3(z,-z)}\right] A_{22}(z)\bar{A}_{22}(z)|0\rangle \\
&= -\frac{e^{ip(z)/2}+e^{-ip(z)/2}}{2}\prod_{k=1}^{N} a_5(z,z_k)a_6(z_k,-z)|0\rangle\,. &(3.35)
\end{aligned}$$

From the Yang-Baxter relation (3.23), we also obtain

$$\begin{aligned}
&\left[C_4(z)\bar{E}_1(z) + a_{11}(z,-z)C_5(z)\bar{E}_2(z) + a_{10}(z,-z)C_3(z)\bar{F}(z)\right]|0\rangle \\
&= \left[\frac{a_{13}(z,-z)a_{10}(z,-z)}{a_3(z,-z)}B(z)\bar{B}(z) + \frac{a_{10}(z,-z)a_{14}(z,-z)}{a_3(z,-z)}A_{11}(z)\bar{A}_{11}(z) - a_{10}(z,-z)D(z)\bar{D}(z)\right]|0\rangle\,, \\
&\left[a_{11}(z,-z)C_4(z)\bar{E}_1(z) + C_5(z)\bar{E}_2(z) + a_{10}(z,-z)C_3(z)\bar{F}(z)\right]|0\rangle \\
&= \left[\frac{a_{13}(z,-z)a_{10}(z,-z)}{a_3(z,-z)}B(z)\bar{B}(z) + \frac{a_{10}(z,-z)a_{14}(z,-z)}{a_3(z,-z)}A_{22}(z)\bar{A}_{22}(z) - a_{10}(z,-z)D(z)\bar{D}(z)\right]|0\rangle\,, \\
&\left[a_9(z,-z)C_4(z)\bar{E}_1(z) + a_9(z,-z)C_5(z)\bar{E}_2(z) - C_3(z)\bar{F}(z)\right]|0\rangle \\
&= \left[-a_{13}(z,-z)B(z)\bar{B}(z) + a_{13}(z,-z)D(z)\bar{D}(z)\right]|0\rangle\,. \qquad (3.36)
\end{aligned}$$

Using the definition of $\tilde{\mathcal{A}}_{ij}(z)$, we obtain

$$\begin{aligned}
\mathcal{D}(z)|0\rangle &= \frac{a_{10}(z,-z)}{a_3(z,-z)}\left[\tilde{\mathcal{A}}_{11}(z)+\tilde{\mathcal{A}}_{22}(z)\right]|0\rangle + \frac{a_{13}(z,-z)}{a_3(z,-z)}\mathcal{B}(z)|0\rangle \\
&\quad + \frac{a_{14}(z,-z)}{a_3(z,-z)}D(z)\bar{D}(z)|0\rangle\,. \qquad (3.37)
\end{aligned}$$

Defining

$$\tilde{\mathcal{D}}(z) = \mathcal{D}(z) - \frac{a_{10}(z,-z)}{a_3(z,-z)}\left[\tilde{\mathcal{A}}_{11}(z)+\tilde{\mathcal{A}}_{22}(z)\right] - \frac{a_{13}(z,-z)}{a_3(z,-z)}\mathcal{B}(z)\,, \qquad (3.38)$$

we obtain

$$\tilde{\mathcal{D}}(z)|0\rangle = \frac{a_{14}(z,-z)}{a_3(z,-z)}D(z)\bar{D}(z)|0\rangle\,. \qquad (3.39)$$

The transfer matrix (2.26) can be expressed in terms of elements of the double-row monodromy matrix (3.16)

$$\begin{aligned} t(z) &= \mathrm{str}_a \left\{ R^L_a(z)\, U_a(z) \right\} \qquad (3.40) \\ &= K^+_1(z)\mathcal{A}_{11}(z) + K^+_2(z)\mathcal{A}_{21}(z) + K^+_3(z)\mathcal{A}_{12}(z) + K^+_4(z)\mathcal{A}_{22}(z) - \mathcal{D}(z) - \mathcal{B}(z) \\ &= \bar{K}^+_1(z)\tilde{\mathcal{A}}_{11}(z) + \bar{K}^+_2(z)\tilde{\mathcal{A}}_{21}(z) + \bar{K}^+_3(z)\tilde{\mathcal{A}}_{12}(z) + \bar{K}^+_4(z)\tilde{\mathcal{A}}_{22}(z) + \bar{K}^+_5(z)\tilde{\mathcal{D}}(z) + \bar{K}^+_6(z)\mathcal{B}(z)\,, \end{aligned}$$

where

$$\begin{aligned} &\bar{K}^+_2(z) = K^+_2(z), \quad \bar{K}^+_3(z) = K^+_3(z), \quad \bar{K}^+_5(z) = -1\,, \\ &\bar{K}^+_1(z) = K^+_1(z) - \frac{a_{10}(z,-z)}{a_3(z,-z)}\,, \quad \bar{K}^+_4(z) = K^+_4(z) - \frac{a_{10}(z,-z)}{a_3(z,-z)}\,, \\ &\bar{K}^+_6(z) = -1 - \frac{a_9(z,-z)}{a_3(z,-z)}\left[K^+_1(z) + K^+_4(z)\right] - \frac{a_{13}(z,-z)}{a_3(z,-z)} = -a_{12}(z,-z)\,. \qquad (3.41) \end{aligned}$$

It is now straightforward to verify from the above results that the reference state is an eigenstate of the transfer matrix, with eigenvalue

$$\begin{aligned} \Lambda_0(z) &= \bar{K}^+_6(z)\prod_{k=1}^{N} a_3(z,z_k)a_3(z_k,-z) + \bar{K}^+_5(z)a_{14}(z,-z)\prod_{k=1}^{N} a_{14}(z,z_k)a_{14}(z_k,-z) \\ &+ 2\cos(2\theta)\cos^2\left[\frac{p(z)}{2}\right]\prod_{k=1}^{N} a_5(z,z_k)a_6(z_k,-z)\,. \qquad (3.42) \end{aligned}$$

3.1.2 The action of the transfer matrix on the first-level eigenstates

Using the right reflection equation for the double-row monodromy matrix (c.f. (2.22))

$$S_{12}(z_1,z_2)\, U_1(z_1)\, S_{21}(z_2,-z_1)\, U_2(z_2) = U_2(z_2)\, S_{12}(z_1,-z_2)\, U_1(z_1)\, S_{21}(-z_2,-z_1)\,, \qquad (3.43)$$

and the definitions of $\tilde{\mathcal{A}}_{ij}(z)$ and $\tilde{\mathcal{D}}(z)$, we obtain – after lengthy computations – the following exchange relations:

$$\mathcal{B}(z_1)\mathcal{B}_k(z_2) = \frac{a_3(z_2,z_1)a_6(z_1,-z_2)}{a_3(z_2,-z_1)a_6(-z_1,-z_2)}\mathcal{B}_k(z_2)\mathcal{B}(z_1) + \text{u.t.}, \tag{3.44}$$

$$\tilde{\mathcal{A}}_{a_1d_1}(z_1)\mathcal{B}_{c_1}(z_2) = \frac{r(z_1,-z_2)^{a_2b_1}_{a_1c_2}\bar{r}(-z_2,-z_1)^{d_1c_1}_{d_2b_1}}{a_5(z_1,z_2)a_6(z_2,-z_1)}\mathcal{B}_{c_2}(z_2)\tilde{\mathcal{A}}_{a_2d_2}(z_1) + \text{u.t.}, \tag{3.45}$$

$$\tilde{\mathcal{D}}(z_1)\mathcal{B}_k(z_2) = \frac{a_{12}(z_1,-z_2)a_5(-z_2,-z_1)}{a_{14}(z_1,z_2)a_5(z_1,-z_2)}\mathcal{B}_k(z_2)\tilde{\mathcal{D}}(z_1) + \text{u.t.}, \tag{3.46}$$

$$\begin{aligned}\vec{\mathcal{B}}(z_1)\otimes\vec{\mathcal{B}}(z_2) = &-\frac{a_8(-z_2,-z_1)}{a_{14}(-z_2,-z_1)a_6(z_2,-z_1)}\left[a_{14}(z_1,-z_2)\mathcal{F}(z_2)\mathcal{B}(z_1) - a_{14}(z_2,-z_1)\mathcal{F}(z_1)\mathcal{B}(z_2)\right]\vec{\xi} \\ &-\frac{a_6(z_1,-z_2)}{a_3(z_1,z_2)a_6(z_2,-z_1)}\left\{\vec{\mathcal{B}}(z_2)\otimes\vec{\mathcal{B}}(z_1) + \frac{a_8(z_1,-z_2)}{a_6(z_1,-z_2)}\mathcal{F}(z_2)\,\vec{\xi}\cdot[I\otimes\mathcal{A}(z_1)]\right\}\cdot\tilde{r}(-z_2,-z_1) \\ &-\frac{a_8(z_2,-z_1)}{a_6(z_2,-z_1)}\mathcal{F}(z_1)\,\vec{\xi}\cdot[I\otimes\mathcal{A}(z_2)]\ ,\end{aligned} \tag{3.47}$$

where

$$\vec{\xi} = \begin{pmatrix} 0, & 1, & -1, & 0 \end{pmatrix}\ , \tag{3.48}$$

$$\mathcal{A}(z) = \begin{pmatrix} \mathcal{A}_{11}(z) & \mathcal{A}_{12}(z) \\ \mathcal{A}_{21}(z) & \mathcal{A}_{22}(z) \end{pmatrix}\ , \qquad \vec{\mathcal{B}}(z) = (\mathcal{B}_1(z)\,,\mathcal{B}_2(z))\ , \tag{3.49}$$

$$r(z_1,z_2) = \begin{pmatrix} h_1(z_1,z_2) & & & \\ & h_2(z_1,z_2) & h_3(z_1,z_2) & \\ & h_3(z_1,z_2) & h_2(z_1,z_2) & \\ & & & h_1(z_1,z_2) \end{pmatrix}\ , \tag{3.50}$$

$$\bar{r}(z_1,z_2) = \begin{pmatrix} h_4(z_1,z_2) & & & \\ & h_5(z_1,z_2) & h_6(z_1,z_2) & \\ & h_6(z_1,z_2) & h_5(z_1,z_2) & \\ & & & h_4(z_1,z_2) \end{pmatrix}\ , \tag{3.51}$$

$$\tilde{r}(z_1,z_2) = \begin{pmatrix} h_4(z_1,z_2) & & & \\ & h_6(z_1,z_2) & h_5(z_1,z_2) & \\ & h_5(z_1,z_2) & h_6(z_1,z_2) & \\ & & & h_4(z_1,z_2) \end{pmatrix}\ , \tag{3.52}$$

with

$$\begin{aligned} &h_1(z_1,z_2) = a_1(z_1,z_2) + \frac{a_9(z_1,z_2)a_{10}(z_1,z_2)}{a_3(z_1,z_2)}, \quad h_2(z_1,z_2) = a_{12}(z_1,z_2)\,, \\ &h_3(z_1,z_2) = h_1(z_1,z_2) - h_2(z_1,z_2)\,, \quad h_4(z_1,z_2) = a_1(z_1,z_2)\,, \\ &h_5(z_1,z_2) = a_{12}(z_1,z_2) - \frac{a_7(z_1,z_2)a_8(z_1,z_2)}{a_{14}(z_1,z_2)}, \quad h_6(z_1,z_2) = h_4(z_1,z_2) - h_5(z_1,z_2)\,, \end{aligned} \tag{3.53}$$

and "u.t." denotes so-called unwanted terms, which we do not explicitly write.

In terms of $u(z)$ (2.9), we can now write

$$
\begin{aligned}
r(z_1,-z_2) &= -ih_3(z_1,-z_2)\, R^{(2)}(u_1+u_2-i)\,, \\
\bar{r}(-z_2,-z_1) &= -ih_6(-z_2,-z_1)\, R^{(2)}(u_1-u_2)\,,
\end{aligned} \tag{3.54}
$$

where $u_j \equiv u(z_j)$, and $R^{(2)}(u)$ is the familiar spin-1/2 XXX R-matrix

$$R^{(2)}(u) = u\mathbb{I} + i\Pi\,, \tag{3.55}$$

where $\mathbb{I}$ and Π are the 4×4 identity and permutation matrices, respectively.

The first-level eigenvectors of the transfer matrix have the general structure [18, 19]

$$|\Phi_M(\lambda_1,\cdots,\lambda_M)\rangle = \vec{\Phi}_M(\lambda_1,\cdots,\lambda_M)\cdot\vec{F}|0\rangle, \tag{3.56}$$

where $\{\lambda_j\}$ are Bethe roots, $\vec{\Phi}_M(\lambda_1,\cdots,\lambda_M)$ are 2^M-dimensional row-vectors whose components are operators, and $\vec{F}$ are c-number coefficients. The $\vec{\Phi}_M(\lambda_1,\cdots,\lambda_M)$ can be shown to satisfy a recursion relation of the form[6]

$$
\begin{aligned}
&\vec{\Phi}_M(\lambda_1,\cdots,\lambda_M) = \vec{\mathcal{B}}(\lambda_1)\otimes\vec{\Phi}_{M-1}(\lambda_2,\cdots,\lambda_M) \qquad (3.57)\\
&\quad+\sum_{j=2}^{M}\left[\vec{\xi}\otimes\mathcal{F}(\lambda_1)\vec{\Phi}_{M-2}(\lambda_2,\cdots,\lambda_{j-1},\lambda_{j+1},\cdots,\lambda_M)\,\mathcal{B}(\lambda_j)\right] g_{j-1}^{(M)}(\lambda_1,\cdots,\lambda_M)\\
&\quad-\sum_{j=2}^{M}\mathcal{F}(\lambda_1)\vec{\Phi}_{M-2}(\lambda_2,\cdots,\lambda_{j-1},\lambda_{j+1},\cdots,\lambda_M)\otimes\left[\vec{\xi}\cdot(I\otimes\tilde{\mathcal{A}}(\lambda_j))\right]\, h_{j-1}^{(M)}(\lambda_1,\cdots,\lambda_M)\,,
\end{aligned}
$$

for certain functions $g_{j-1}^{(M)}$ and $h_{j-1}^{(M)}$ whose explicit expressions will not be needed here, and $\vec{\Phi}_0 = 1$. In particular, $\vec{\Phi}_1(\lambda) = \vec{\mathcal{B}}(\lambda)$.

Let us define

$$y_j = x^-(\lambda_j)\,, \qquad \tilde{u}_j = \tfrac{g}{2}(y_j + \tfrac{1}{y_j}) + \tfrac{i}{2}\,, \qquad j = 1,\cdots,M\,. \tag{3.58}$$

By using the exchange relations (3.44)-(3.46) and the values of $\tilde{\mathcal{A}}_{ij}(z)$, $\tilde{\mathcal{D}}(z)$ and $\mathcal{B}(z)$ when acting on the reference state, we obtain

$$
\begin{aligned}
t(z)|\Phi_M(\lambda_1,\cdots,\lambda_M)\rangle &= \Bigg\{\bar{K}_6^+(z)\prod_{j=1}^{M}\frac{a_3(\lambda_j,z)a_6(z,-\lambda_j)}{a_3(\lambda_j,-z)a_6(-z,-\lambda_j)}\prod_{k=1}^{N}a_3(z,z_k)a_3(z_k,-z)\\
&+\bar{K}_5^+(z)a_{14}(z,-z)\prod_{j=1}^{M}\frac{a_{12}(z,-\lambda_j)a_5(-\lambda_j,-z)}{a_{14}(z,\lambda_j)a_5(z,-\lambda_j)}\prod_{k=1}^{N}a_{14}(z,z_k)a_{14}(z_k,-z)\\
&+\cos^2\left[\frac{p(z)}{2}\right]\prod_{j=1}^{M}-\frac{h_3(z,-\lambda_j)h_6(-\lambda_j,-z)}{a_5(z,\lambda_j)a_6(\lambda_j,-z)}\prod_{k=1}^{N}a_5(z,z_k)a_6(z_k,-z)\,t^{(2)}(z)\Bigg\}\\
&\times|\Phi_M(\lambda_1,\cdots,\lambda_M)\rangle + \text{u.t.}\,,
\end{aligned} \tag{3.59}
$$

[6] We note a typo in the third term of Eq. (3.40) in [19], and we thank X.-W. Guan for correspondence on this point.

where $t^{(2)}(z)$ is the second-level nested transfer matrix with inhomogeneities $\{\tilde{u}_j\}$

$$t^{(2)}(z) = \mathrm{tr}_0\{K_0^{(2)+}(u)\, T_0^{(2)}(u, \{\tilde{u}_j\})\, K_0^{(2)-}(u)\, \widehat{T}_0^{(2)}(u, \{\tilde{u}_j\})\}\,, \tag{3.60}$$

with

$$T_0^{(2)}(u, \{\tilde{u}_j\}) = R_{0,1}^{(2)}(u + \tilde{u}_1 - i) \cdots R_{0,M}^{(2)}(u + \tilde{u}_M - i)\,, \tag{3.61}$$

$$\widehat{T}_0^{(2)}(u, \{\tilde{u}_j\}) = R_{M,0}^{(2)}(u - \tilde{u}_M) \cdots R_{1,0}^{(2)}(u - \tilde{u}_1)\,, \tag{3.62}$$

$$K^{(2)-}(u) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.63}$$

$$K^{(2)+}(u) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}. \tag{3.64}$$

We remind the reader that $u = u(z)$ is given by (2.9), and therefore

$$u(z) = \tfrac{g}{4}\left[x^+(z) + \tfrac{1}{x^+(z)} + x^-(z) + \tfrac{1}{x^-(z)}\right]. \tag{3.65}$$

3.2 Off-diagonal Bethe ansatz

In view of (3.59), in order to determine the eigenvalues of the transfer matrix (2.26), it now remains to diagonalize the nested transfer matrix $t^{(2)}(z)$. We recognize the latter as the transfer matrix of an open spin-1/2 XXX chain of length M with non-diagonal boundary terms. Therefore, using the off-diagonal Bethe ansatz [20, 21] (see also [33]), we can immediately write down an expression for the corresponding eigenvalues.

Indeed, let us introduce the following functions

$$a^{(2)}(u) = \frac{2u - i}{2u} \prod_{j=1}^{M} (u - \tilde{u}_j + i)(u + \tilde{u}_j)\,, \tag{3.66}$$

$$d^{(2)}(u) = \frac{2u + i}{2u} \prod_{j=1}^{M} (u - \tilde{u}_j)(u + \tilde{u}_j - i)\,. \tag{3.67}$$

According to the off-diagonal Bethe ansatz, the eigenvalues of $t^{(2)}(z)$ can be given by

$$\begin{aligned} \Lambda^{(2)}(z) &= a^{(2)}(u) \frac{Q_2(u - i)}{Q_2(u)} + d^{(2)}(u) \frac{Q_2(u + i)}{Q_2(u)} \\ &+ \frac{2\left[\cos(2\theta) - 1\right]}{Q_2(u)} \prod_{j=1}^{M} (u - \tilde{u}_j)(u + \tilde{u}_j - i)(u - \tilde{u}_j + i)(u + \tilde{u}_j)\,, \end{aligned} \tag{3.68}$$

where the polynomial $Q_2(u)$ is parameterized by M Bethe roots $\{w_j\}$

$$Q_2(u) = \prod_{j=1}^{M} (u - w_j)(u + w_j)\,. \tag{3.69}$$

One can recognize (after multiplying both sides by $Q_2(u)$) that (3.68) is a T-Q equation with an additional inhomogeneous term.

3.3 Eigenvalues and Bethe equations

Combining the results (3.59) and (3.68), we conclude that the eigenvalues of the transfer matrix $t(z)$ (2.26) are given by

$$\begin{aligned}
\Lambda(z) &= -a_{12}(z,-z)e^{-iMp(z)}\prod_{j=1}^{M}\frac{[x^{+}(z)-y_j][x^{+}(z)+y_j]}{[x^{-}(z)-y_j][x^{-}(z)+y_j]}\prod_{k=1}^{N}a_3(z,z_k)a_3(z_k,-z)\\
&\quad -a_{14}(z,-z)e^{iMp(z)}\prod_{j=1}^{M}\frac{[x^{-}(z)y_j-1][x^{-}(z)y_j+1]}{[x^{+}(z)y_j-1][x^{+}(z)y_j+1]}\prod_{k=1}^{N}a_{14}(z,z_k)a_{14}(z_k,-z)\\
&\quad +\cos^2\left[\frac{p(z)}{2}\right]\prod_{k=1}^{N}a_5(z,z_k)a_6(z_k,-z)\Bigg\{ \qquad (3.70)\\
&\quad \frac{2u-i}{2u}e^{-iMp(z)}\frac{Q_2(u-i)}{Q_2(u)}\prod_{j=1}^{M}\frac{[x^{+}(z)-y_j][x^{+}(z)+y_j]}{[x^{-}(z)-y_j][x^{-}(z)+y_j]}\\
&\quad +\frac{2u+i}{2u}e^{iMp(z)}\frac{Q_2(u+i)}{Q_2(u)}\prod_{j=1}^{M}\frac{[x^{-}(z)y_j-1][x^{-}(z)y_j+1]}{[x^{+}(z)y_j-1][x^{+}(z)y_j+1]}\\
&\quad +\frac{2[\cos(2\theta)-1]}{Q_2(u)}\prod_{j=1}^{M}(\frac{g^2}{4})\frac{[x^{+}(z)+y_j][x^{+}(z)-y_j][x^{-}(z)y_j+1][x^{-}(z)y_j-1]}{x^{+}(z)x^{-}(z)y_j^2}\Bigg\}.
\end{aligned}$$

The requirement that $\Lambda(z)$ should not have any poles leads to the following Bethe equations[7]

$$\prod_{l=1}^{N}\frac{x^{-}(z_l)+y_j}{x^{+}(z_l)+y_j}\frac{x^{+}(z_l)-y_j}{x^{-}(z_l)-y_j}\frac{Q_2(\tilde{u}_j)}{Q_2(\tilde{u}_j-i)}=1\,, \qquad j=1,\cdots,M\,, \tag{3.71}$$

$$\begin{aligned}
&a^{(2)}(w_k)Q_2(w_k-i)+d^{(2)}(w_k)Q_2(w_k+i) \qquad (3.72)\\
&+2\left[\cos(2\theta)-1\right]\prod_{j=1}^{M}(w_k-\tilde{u}_j)(w_k+\tilde{u}_j-i)(w_k-\tilde{u}_j+i)(w_k+\tilde{u}_j)=0\,, \quad k=1,\cdots,M\,.
\end{aligned}$$

These results can be reexpressed more succinctly by using the shorthand notation of [35, 15]

$$\begin{aligned}
R^{(\pm)}(z) &= \prod_{i=1}^{N}\left(x(z)-x^{\mp}(z_i)\right)\left(x(z)+x^{\pm}(z_i)\right)\,,\\
B^{(\pm)}(z) &= R^{(\pm)}(z+\omega_2)=\prod_{i=1}^{N}\left(\frac{1}{x(z)}-x^{\mp}(z_i)\right)\left(\frac{1}{x(z)}+x^{\pm}(z_i)\right)\,. \qquad (3.73)
\end{aligned}$$

[7]It should also be possible to obtain the Bethe equations from the cancellation of the unwanted terms that appear when the transfer matrix acts on an off-shell Bethe state; however, we have not determined the complete off-shell equation. Such an off-shell equation has been found recently for the XXZ chain [34].

For example, the expression $R^{(-)-}(z)$ should be understood to mean

$$R^{(-)-}(z) \;=\; \prod_{i=1}^{N} \left(x^-(z) - x^+(z_i)\right) \left(x^-(z) + x^-(z_i)\right) .$$

Similarly,

$$\begin{aligned} B_1R_3(z) &= \prod_{j=1}^{M} (x(z) - y_j)\,(x(z) + y_j) \,, \\ R_1B_3(z) &= B_1R_3(z+\omega_2) = \prod_{j=1}^{M} \left(\frac{1}{x(z)} - y_j\right)\left(\frac{1}{x(z)} + y_j\right) . \end{aligned} \tag{3.74}$$

Moreover, if $f(u)$ is any function of u, then $f^{\pm} = f(u \pm \frac{i}{2})\,, f^{\pm\pm} = f(u \pm i)$.

In terms of this notation, the eigenvalues (3.70) are given by

$$\begin{aligned} \Lambda(z) = e^{i(N-M+1)p}\frac{1}{R^{(+)+}B^{(-)+}}\Bigg\{ &-\rho_1 R^{(-)-}B^{(-)+}\frac{(B_1R_3)^+}{(B_1R_3)^-} - \rho_2 R^{(+)-}B^{(+)+}\frac{(R_1B_3)^-}{(R_1B_3)^+} \\ &+\tfrac{1}{2}(\rho_1+\rho_2)R^{(+)-}B^{(-)+}\Bigg[\frac{u^-}{u}\frac{(B_1R_3)^+}{(B_1R_3)^-}\frac{Q_2^{--}}{Q_2} + \frac{u^+}{u}\frac{(R_1B_3)^-}{(R_1B_3)^+}\frac{Q_2^{++}}{Q_2} \\ &+2[\cos(2\theta)-1]\frac{(R_1B_3)^-(B_1R_3)^+}{Q_2}\prod_{j=1}^{M}\left(-\tfrac{g^2}{4y_j^2}\right)\Bigg]\Bigg\}\,, \end{aligned} \tag{3.75}$$

where

$$\rho_1(z) = \frac{(1+(x^-)^2)(x^-+x^+)}{2x^+(1+x^-x^+)}\,, \quad \rho_2(z) = \rho_1(-z-\omega_2) = \frac{x^-(1+(x^+)^2)(x^-+x^+)}{2(x^+)^2(1+x^-x^+)}\,. \tag{3.76}$$

The corresponding Bethe equations are

$$\left.\frac{R^{(-)-}}{R^{(+)-}}\frac{Q_2}{Q_2^{--}}\right|_{z=\lambda_j} = 1\,, \qquad j = 1,\ldots,M\,, \tag{3.77}$$

$$\left.\left[\tfrac{u^-}{u}Q_1^+Q_2^{--} + \tfrac{u^+}{u}Q_1^-Q_2^{++} + 2[\cos(2\theta)-1]\,Q_1^+Q_1^-\right]\right|_{u=w_k} = 0\,, \quad k = 1,\cdots,M\,, \tag{3.78}$$

where [8]

$$Q_1(u) = \prod_{j=1}^{M}(u + \tfrac{i}{2} - \tilde{u}_j)(u - \tfrac{i}{2} + \tilde{u}_j)\,. \tag{3.79}$$

[8]We recall the definitions (3.58) and also note the identity

$$(B_1R_3)^{\pm}(R_1B_3)^{\pm} = Q_1^{\pm}\prod_{j=1}^{M}\left(-\tfrac{4y_j^2}{g^2}\right) .$$

The Bethe equations (3.77) and (3.78) are equivalent to (3.71) and (3.72), respectively.

For $\theta = 0$, the last ("inhomogeneous") term in (3.75) vanishes; and we see (using $\frac{\rho_2}{\rho_1} = \frac{u^+}{u^-}$) that our result (3.75) for the transfer matrix eigenvalue is consistent with the $sl(2)$ grading result (C.8) in [15].

Interestingly, for $\theta = \pi/2$, the inhomogeneous term does *not* vanish, even though the boundary S-matrices are diagonal for this case (see (3.2), (3.3) and (3.64))! This is the price we pay for having an expression for $\Lambda(z)$ that is analytic in θ. An alternative expression is [9]

$$\Lambda(z) = e^{i(N-M+1)p} \frac{1}{R^{(+)+}B^{(-)+}} \Bigg\{ -\rho_1 R^{(-)-}B^{(-)+} \frac{(B_1R_3)^+}{(B_1R_3)^-} - \rho_2 R^{(+)-}B^{(+)+} \frac{(R_1B_3)^-}{(R_1B_3)^+}$$
$$+\tfrac{1}{2}(\rho_1+\rho_2) R^{(+)-}B^{(-)+} \Bigg[s(\theta) \frac{u^-}{u} \frac{(B_1R_3)^+}{(B_1R_3)^-} \frac{Q_2^{--}}{Q_2} + s(\theta) \frac{u^+}{u} \frac{(R_1B_3)^-}{(R_1B_3)^+} \frac{Q_2^{++}}{Q_2}$$
$$+2[\cos(2\theta) - s(\theta)] \frac{(R_1B_3)^-(B_1R_3)^+}{Q_2} \prod_{j=1}^{M} \Big(-\tfrac{g^2}{4y_j^2} \Big) \Bigg] \Bigg\} ,$$

where

$$s(\theta) = \frac{\cos(2\theta)}{|\cos(2\theta)|} = \begin{cases} \;\;1 & 0 \le \theta < \frac{\pi}{4} \\ -1 & \frac{\pi}{4} < \theta \le \frac{\pi}{2} \end{cases} .$$

(For $\theta = \pi/4$, either $s = +1$ or $s = -1$ can be chosen.) The inhomogeneous term in this expression does vanish for both $\theta = 0$ and $\theta = \pi/2$; and for $\theta = \pi/2$, this result for the transfer matrix eigenvalue is consistent with the duality transformation of the result (3.20) in [16].

3.4 Degeneracy and multiplicity

The degeneracy of the transfer matrix eigenvalue (3.75) corresponding to a given solution of the Bethe equations (3.77)-(3.78), as well as the number of such solutions (multiplicity), can be inferred from the unbroken $su(2)$ symmetry (2.29) of the transfer matrix. [10]

Indeed, we expect (see e.g. [18, 36]) that the Bethe states are $su(2)$ lowest-weight states, with

$$s = -m = \tfrac{1}{2}(N-M) , \tag{3.80}$$

where $s(s+1)$ is the eigenvalue of $\vec{S}^2$, and m is the eigenvalue of S^z. Since $s \ge 0$, it follows that M can take the values $0, 1, \ldots, N$. Hence, for given values of N and M, we expect that

[9] This expression corresponds to choosing a different parametrization for the T-Q equation (3.68), with $a^{(2)}(u)$ and $d^{(2)}(u)$ rescaled by $s(\theta)$ and with a corresponding modification of the inhomogeneous term. One can show that both parametrizations satisfy the necessary requirements of crossing symmetry, initial condition, asymptotic behavior, and functional relation [20, 21].

[10] We assume here that there are no pathologies, such as singular solutions of the Bethe equations, or accidental spectrum degeneracy.

the degeneracy $\mathcal{D}(N,M)$ of the corresponding eigenvalue is given by

$$\mathcal{D}(N,M) = 2s+1 = N-M+1\,. \tag{3.81}$$

For one site, the decomposition of the 4-dimensional vector space into $su(2)$ representations is given by $\mathbf{0}\oplus\mathbf{0}\oplus\frac{\mathbf{1}}{\mathbf{2}}$. For N sites, the decomposition of the space of states into a direct sum of $su(2)$ irreducible representations can be easily determined using the Clebsch-Gordan theorem

$$\left(\mathbf{0}\oplus\mathbf{0}\oplus\tfrac{\mathbf{1}}{\mathbf{2}}\right)^{\otimes N} = \bigoplus_{s=0}^{N/2} n_s\, \mathbf{s}\,, \tag{3.82}$$

where n_s is the multiplicity of spin s. With the help of the multinomial theorem, an explicit expression for n_s can be derived

$$n_s = \sum_{\substack{k_1,k_2,k_3=0\\ k_1+k_2+k_3=N}}^{N} \frac{N!}{k_1!k_2!k_3!} d_s(k_3)\,, \tag{3.83}$$

where

$$d_s(k_3) = \binom{k_3}{\frac{k_3}{2}-s} - \binom{k_3}{\frac{k_3}{2}-s-1}\,, \tag{3.84}$$

and $\binom{n}{m} = \frac{n!}{(n-m)!m!}$ is defined to be 0 if m is outside of the interval $[0,n]$ or if m is not an integer. Hence, for given values of N and M, we expect that the number of solutions $\mathcal{N}(N,M)$ of the Bethe equations is given by

$$\mathcal{N}(N,M) = n_s\Big|_{s=\frac{1}{2}(N-M)}\,, \tag{3.85}$$

where n_s is given by (3.83). We have verified that the expressions (3.81) and (3.85) satisfy the completeness constraint

$$\sum_{M=0}^{N} \mathcal{D}(N,M)\,\mathcal{N}(N,M) = 4^N\,. \tag{3.86}$$

3.5 Numerical checks

We have numerically checked our Bethe ansatz solution (3.75)-(3.78), as well as the formulas for degeneracies (3.81) and multiplicities (3.85), for $N=1$ and $N=2$.

For $N=1$, we expect according to (3.85) one solution with $M=0$ (namely, the trivial solution with no Bethe roots, corresponding to the reference state (3.1)), and two solutions with $M=1$. We indeed find these solutions, as shown in Table 1. The corresponding eigenvalues obtained from (3.75) match with the 4 eigenvalues obtained by direct diagonalization of the transfer matrix (2.26).

M	$\{y_j\}$	$\{w_k\}$	degeneracy
0	-	-	2
1	29.67201576134	4.98947370172	1
1	37.59406315269	4.98947370172	1

Table 1: Solutions of the Bethe equations (3.77)-(3.78) and degeneracies (3.81) of the corresponding eigenvalues (3.75) for $N = 1$ with $g = 0.3$, $\theta = 0.7$, $z_1 = 0.1$.

M	$\{y_j\}$	$\{w_k\}$	degeneracy
0	-	-	3
1	1.688644387948	0.522719047641	2
1	5.425599080922	0.735491947795 i	2
1	8.578126668210	1.731865667315	2
1	21.148232916045	2.466473986963	2
2	5.114946745748, 18.101713927816	0.461647669632, 1.336004479377	1
2	2.201092869446, 16.716623804005	0.617999330346, 1.271260542723	1
2	1.035542549912, 7.917752654460	1.033924690882 $\pm$ 0.264550555698 i	1
2	10.157304730304 $\pm$ 2.8436106714245 i	0.880379764515, 1.105874319271	1
2	0.129605338411, 6.858626722293	1.783773124400, 1.088034934890 i	1

Table 2: Solutions of the Bethe equations (3.77)-(3.78) and degeneracies (3.81) of the corresponding eigenvalues (3.75) for $N = 2$ with $g = 0.3$, $\theta = 0.7$, $z_1 = 0.8$, $z_2 = 0.4$.

Similarly, for $N = 2$, we expect (3.85) one solution with $M = 0$, four solutions with $M = 1$, and five solutions with $M = 2$. We indeed find these solutions, as shown in Table 2. The corresponding eigenvalues obtained from (3.75) match with the 16 eigenvalues obtained by direct diagonalization of the transfer matrix.

In short, we have verified that our Bethe ansatz solution correctly gives the complete set of eigenvalues of the transfer matrix for $N = 1$ and $N = 2$.

4 Discussion

For the transfer matrix (2.26) of the $Y_\theta - Y$ system, we have determined the exact eigenvalues (3.75) in terms of solutions of a corresponding set of Bethe equations (3.77)-(3.78). We have checked this result numerically for small system size.

The $Y_\theta = 0$ boundary S-matrix (2.23) is one of the few known integrable AdS/CFT boundary S-matrices with a free parameter. (Other examples are discussed in [32, 37].) The present work represents the first time in the AdS/CFT context that an open-chain transfer matrix with a non-diagonal boundary S-matrix is diagonalized.

We hope to use this result in a future publication to compute asymptotic energies and finite-size corrections for one-particle states, as a function of the angle θ. Such corrections have already been computed for the special (diagonal) cases $Y - Y$ ($\theta = 0$) and $\bar{Y} - Y$ ($\theta = \pi/2$) in [14, 15] and [16], respectively. The latter system is noteworthy for the presence of tachyons in its spectrum.

We expect that similar techniques can also be used to analyze other integrable cases with non-diagonal boundary S-matrices.

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A Generating functional for higher transfer matrices

In the body of this paper, we have focused on a transfer matrix $t(z)$ (2.26) whose auxiliary space belongs to the fundamental (4-dimensional) representation of $su(2|2)$. This transfer matrix is only the first member of an infinite hierarchy of commuting transfer matrices $T_{a,s}$ (with $T_{1,1} = t(z)$) whose auxiliary spaces belong to rectangular representations of $su(2|2)$, and which satisfy the Hirota equation [35]

$$T^{+}_{a,s}\, T^{-}_{a,s} = T_{a+1,s}\, T_{a-1,s} + T_{a,s+1}\, T_{a,s-1}\,. \tag{A.1}$$

We propose here a generating functional for the eigenvalues of these transfer matrices (which we also denote by $T_{a,s}$), which are useful for computing finite-size corrections (see e.g. [35, 15, 16]). This generating functional is a generalization of the one proposed in [33] for the XXX chain with nondiagonal boundary terms.

In order to streamline the notation, we rewrite the eigenvalue result (3.75) as

$$T_{1,1} = h\, \hat{T}_{1,1}\,, \qquad \hat{T}_{1,1} = -A - B + G + H + C\,, \tag{A.2}$$

where h is a normalization factor

$$h = \rho_1 \left(\frac{x^+}{x^-}\right)^{N-M+1} \frac{R^{(+)-}}{R^{(+)+}}\,, \tag{A.3}$$

and

$$
\begin{aligned}
A &= \frac{R^{(-)-}}{R^{(+)-}} \frac{(B_1R_3)^+}{(B_1R_3)^-}\,, \\
B &= \frac{u^+}{u^-} \frac{B^{(+)+}}{B^{(-)+}} \frac{(R_1B_3)^-}{(R_1B_3)^+}\,, \\
G &= \frac{(B_1R_3)^+}{(B_1R_3)^-} \frac{Q_2^{--}}{Q_2}\,, \\
H &= \frac{u^+}{u^-} \frac{(R_1B_3)^-}{(R_1B_3)^+} \frac{Q_2^{++}}{Q_2}\,, \\
C &= [\cos(2\theta) - 1] \left(1 + \frac{u^+}{u^-}\right) \frac{(R_1B_3)^-(B_1R_3)^+}{Q_2} \prod_{j=1}^{M} \left(-\tfrac{g^2}{4y_j^2}\right)\,.
\end{aligned} \tag{A.4}
$$

We propose that the generating functional for antisymmetric representations is given by

$$
\begin{aligned}
W^{-1} &= (1 - \mathcal{D}A\mathcal{D})^{-1} \left[1 - \mathcal{D}(G + H + C)\mathcal{D} + \mathcal{D}G\mathcal{D}^2H\mathcal{D}\right] (1 - \mathcal{D}B\mathcal{D})^{-1} \\
&= \sum_{a=0}^{\infty} (-1)^a \mathcal{D}^a \, \hat{T}_{a,1} \, \mathcal{D}^a\,,
\end{aligned} \tag{A.5}
$$

where $\mathcal{D} = e^{-\frac{i}{2}\partial_u}$ implying $\mathcal{D}f = f^-\mathcal{D}$, with

$$
T_{a,1} = h^{[a-1]} h^{[a-3]} \cdots h^{[3-a]} h^{[1-a]} \, \hat{T}_{a,1}\,, \tag{A.6}
$$

where $f^{[\pm n]} = f(u \pm \frac{in}{2})$. By expanding both sides of (A.5), we obtain: $\hat{T}_{0,1} = 1$, the result in (A.2) for $\hat{T}_{1,1}$, and

$$
\hat{T}_{2,1} = G^+H^- - A^+ \left(G^- + H^- + C^- - A^-\right) - \left(G^+ + H^+ + C^+ - A^+ - B^+\right) B^-\,, \tag{A.7}
$$

etc.

As a check on our proposal, we observe that for $\theta = 0$ (and therefore $C = 0$), the factor in square brackets in (A.5) factorizes

$$
1 - \mathcal{D}(G + H)\mathcal{D} + \mathcal{D}G\mathcal{D}^2H\mathcal{D} = (1 - \mathcal{D}G\mathcal{D})(1 - \mathcal{D}H\mathcal{D})\,, \tag{A.8}
$$

and therefore the generating functional (A.5) reduces to

$$
W^{-1}\Big|_{\theta=0} = (1 - \mathcal{D}A\mathcal{D})^{-1}(1 - \mathcal{D}G\mathcal{D})(1 - \mathcal{D}H\mathcal{D})(1 - \mathcal{D}B\mathcal{D})^{-1}\,, \tag{A.9}
$$

which coincides with the result (C.10) in [15].

A further check on our proposal is provided by the special case $N = M = 0$ and generic angle θ. For this case, the expressions in (A.4) reduce to

$$
A = 1\,, \quad B = \frac{u^+}{u^-}\,, \quad G = 1\,, \quad H = \frac{u^+}{u^-}\,, \quad C = [\cos(2\theta) - 1] \left(1 + \frac{u^+}{u^-}\right)\,, \tag{A.10}
$$

and hence

$$G + H + C = \cos(2\theta)\left(1 + \frac{u^+}{u^-}\right) . \tag{A.11}$$

The generating functional (A.5) therefore reduces to

$$W^{-1}\Big|_{N=M=0} = (1-\mathcal{D}^2)^{-1}\left[1 - \cos(2\theta)\mathcal{D}(1+\tfrac{u^+}{u^-})\mathcal{D} + \mathcal{D}^3\tfrac{u^+}{u^-}\mathcal{D}\right](1-\mathcal{D}\tfrac{u^+}{u^-}\mathcal{D})^{-1} , \tag{A.12}$$

which coincides with the result given by (E.13) and (E.17) in [16]. Moreover, for $N \neq 0$ but still $M = 0$, we obtain

$$W^{-1}\Big|_{M=0} = (1-\mathcal{D}^2\tfrac{R^{(-)}}{R^{(+)}})^{-1}\left[1 - \cos(2\theta)\mathcal{D}(1+\tfrac{u^+}{u^-})\mathcal{D} + \mathcal{D}^3\tfrac{u^+}{u^-}\mathcal{D}\right](1-\tfrac{B^{(+)}}{B^{(-)}}\mathcal{D}\tfrac{u^+}{u^-}\mathcal{D})^{-1} , \tag{A.13}$$

which coincides with (E.21) in [16].

The generating functional for symmetric representations is given by the inverse of (A.5),

$$\begin{aligned} W &= (1-\mathcal{D}B\mathcal{D})\left[1-\mathcal{D}(G+H+C)\mathcal{D} + \mathcal{D}G\mathcal{D}^2 H\mathcal{D}\right]^{-1}(1-\mathcal{D}A\mathcal{D}) \\ &= \sum_{s=0}^{\infty} \mathcal{D}^s\,\hat{T}_{1,s}\,\mathcal{D}^s , \end{aligned} \tag{A.14}$$

with

$$T_{1,s} = h^{[s-1]}h^{[s-3]}\cdots h^{[3-s]}h^{[1-s]}\,\hat{T}_{1,s} . \tag{A.15}$$

By expanding both sides of (A.14), we obtain: $\hat{T}_{1,0} = 1$, the result in (A.2) for $\hat{T}_{1,1}$, and

$$\hat{T}_{1,2} = \left(G^+ + H^+ + C^+ - B^+\right)\left(G^- + H^- + C^- - A^-\right) - G^+H^- , \tag{A.16}$$

etc. As a consistency check, it is now straightforward to verify the Hirota equation (A.1) (which holds also for the renormalized quantities $\hat{T}_{a,s}$) with $a = s = 1$ using the results (A.2), (A.7) and (A.16).

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